

# Remarks on Haar meager sets and Haar null sets in spaces of sequences

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## Abstract

In the paper we will show how to construct a Haar meager set (consequently meager) which is not Haar null, and conversely, a meager Haar null set which is not Haar meager in spaces of sequences:  $l_p$  with  $p \geq 1$ ,  $c_0$  or  $c$ . It refers to the paper [2].

*Keywords:* Haar meager set, Haar null set, meager set  
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## 1. Introduction

In 1972 J.P.R. Christensen defined "Haar null" sets in an abelian Polish group (a topological abelian group with a complete separable metric) in such a way that in a locally compact group it is equivalent to the notion of Haar measure zero set. More precisely, in a fixed abelian Polish group  $X$  a set  $A \subset X$  is called *Haar null* if there is a Borel probability measure  $\mu$  on  $X$  and a Borel set  $B \subset X$  such that  $A \subset B$  and  $\mu(x + B) = 0$  for all  $x \in X$ . These definition has been extended further to nonabelian groups by J. Mycielski [7]. Unaware of the result of Christensen, B.R. Hunt, T. Sauer and J.A. Yorke [3]-[4] found this notation again, but in a topological abelian group with a complete metric (not necessary separable).

In 2013 U.B. Darji introduced another family of "small" sets in an abelian Polish group, which is equivalent to the notion of meager sets in a locally compact group. In an abelian Polish group  $X$  he called a set  $A \subset X$  *Haar meager* if there is a Borel set  $B \subset X$  with  $A \subset B$ , a compact metric space  $K$  and a continuous function  $f : K \rightarrow X$  such that  $f^{-1}(B + x)$  is meager in  $K$  for all  $x \in X$ .

The main aim of the paper is to show easy constructions of a Haar meager, but not Haar null set and, conversely, a meager Haar null set which is not Haar meager in spaces of sequences:  $l_p$  with  $p \geq 1$ ,  $c_0$  or  $c$ .

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## 2. The main results

**Definition 1.** Let  $X$  be an abelian Polish group,  $\mathcal{B}(X)$  be the Borel  $\sigma$ -algebra on  $X$  and denote by  $\mathcal{F}(X)$  the family of all sets  $A \subset X$  such that

$$\forall_{K \subset X \text{-compact}} \exists_{x_K \in X} K + x_K \subset A.$$

In fact  $\mathcal{F}(X)$  is a proper linearly invariant  $\sigma$ -filter. What is interesting, each set  $A \in \mathcal{F}(X) \cap \mathcal{B}(X)$  is neither Haar null (in view of the Ulam theorem), nor Haar meager.

S. Solecki [8], and also E. Matoušková and M. Zelený [5], showed how to find a closed nowhere dense set from the family  $\mathcal{F}$  in any abelian non-locally compact Polish group. We use this fact to construct a Haar meager, but not Haar null set, as well as a meager Haar null set which is not Haar meager in spaces of sequences:  $l_p$  with  $p \geq 1$ ,  $c_0$  or  $c$ .

First we prove two theorems, which we will use in further considerations.

**Theorem 1.** *Let  $X, Y$  be an abelian Polish group. If  $A \subset X$  is Haar meager and  $B \subset B_0$  for some  $B_0 \in \mathcal{B}(Y)$ , then the set  $A \times B \subset X \times Y$  is Haar meager.*

*Proof.* Assume that  $A \subset X$  is Haar meager in  $X$ , i.e. there are a set  $A_0 \in \mathcal{B}(X)$  with  $A \subset A_0$ , a compact metric space  $K$  and a continuous function  $f : K \rightarrow X$  such that  $f^{-1}(A_0 + x)$  is meager in  $K$  for each  $x \in X$ . Take any compact set  $L \subset Y$  and define a continuous function  $g : K \times L \rightarrow X \times Y$  in the following way:

$$g(k, l) = (f(k), l) \text{ for every } (k, l) \in K \times L.$$

Then,

$$\begin{aligned} g^{-1}((A_0 \times B_0) + (x, y)) &= g^{-1}((A_0 + x) \times (B_0 + y)) \\ &= f^{-1}(A_0 + x) \times [(B_0 + y) \cap L] \end{aligned}$$

for each  $(x, y) \in X \times Y$ . Since the set  $f^{-1}(A_0 + x)$  is meager in  $K$ , by the Kuratowski-Ulam theorem the set  $g^{-1}((A_0 \times B_0) + (x, y))$  is meager in  $K \times L$ . Clearly,  $A \times B \subset A_0 \times B_0$  and  $A_0 \times B_0 \in \mathcal{B}(X \times Y)$ , what ends the proof.

**Theorem 2.** *Let  $X, Y$  be an abelian Polish group. For every set  $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$  and non-Haar meager  $B \in \mathcal{B}(Y)$  the set  $A \times B \subset X \times Y$  is not Haar meager.*

*Proof.* Clearly,  $A \times B \in \mathcal{B}(X \times Y)$ . Take a compact metric space  $K$  and a continuous function  $f : K \rightarrow X \times Y$ . Then there are continuous functions  $f_X : K \rightarrow X$  and  $f_Y : K \rightarrow Y$  such that

$$f(z) = (f_X(z), f_Y(z)) \text{ for each } z \in K.$$

The set  $f_X(K)$  is compact in  $X$  and  $A \in \mathcal{F}(X)$ , so  $A \supset f_X(K) + x_K$  for some  $x_K \in X$ . Hence  $f_X^{-1}(A - x_K) \supset K$ . Since  $B$  is not Haar meager in  $Y$ ,  $f_Y^{-1}(B + y_K)$  is comeager in  $K$  for some  $y_K \in Y$ . Moreover:

$$\begin{aligned} f^{-1}((A \times B) + (-x_K, y_K)) &= f^{-1}((A - x_K) \times (B + y_K)) \\ &= f_X^{-1}(A - x_K) \cap f_Y^{-1}(B + y_K) \supset K \cap f_Y^{-1}(B + y_K). \end{aligned}$$

Thus  $f^{-1}((A \times B) + (-x_K, y_K))$  is comeager in  $K$  and, consequently, the set  $A \times B$  is not Haar meager in  $X \times Y$ .

The above theorem suggest the following

**Problem 1.** Let  $X, Y$  be an abelian Polish group. Is it true or false that for every non-Haar meager sets  $A \in \mathcal{B}(X)$  and  $B \in \mathcal{B}(Y)$  the set  $A \times B \subset X \times Y$  is not Haar meager?

A negative answer implies the same question under additional assumption that one of abelian Polish group  $X, Y$  is locally compact.

### 3. Applications

Now, consider the space  $X$  as one of the following spaces of sequences:  $c_0, c$  or  $l_p$  with  $p \geq 1$ . Such spaces have a very nice property:  $X = \mathbb{R} \times X$ .

Fix  $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$ . Let  $S := B \times A \subset \mathbb{R} \times X$ , where  $B \subset \mathbb{R}$  is a meager set of positive Lebesgue measure.

By the analogue of the Fubini theorem [1, Theorem 6] it is easy to observe that  $S$  is not Haar null, since  $S(a) = B$  is the set of the positive Lebesgue measure for each  $a \in A$  and  $A$  is not Haar null. Moreover, in view of Theorem 1, the set  $S$  is Haar meager. In this way we constructed the set  $S$ , which is **Haar meager** (consequently **meager**), but **not Haar null** in  $X$ .

Now, fix  $A \in \mathcal{B}(X) \cap \mathcal{F}(X)$  once again. Let  $T := C \times A \subset \mathbb{R} \times X$ , where  $C \subset \mathbb{R}$  is a comeager set of the Lebesgue measure zero.

By an analogue of the Fubini theorem [1, Theorem 6] we can easily deduce that  $T$  is Haar null in  $X$ , because  $T(a) = C$  is the set of the Lebesgue measure zero for each  $a \in A$  and  $T(a) = \emptyset$  for each  $a \in X \setminus A$ . Moreover, by Theorem 2 the set  $T$  is not Haar meager in  $\mathbb{R} \times X = X$ . Finally,  $T$  is meager according to the Kuratowski-Ulam theorem, because the set  $A$  is meager. In this way we constructed the set  $T$ , which is **Haar null**, **meager**, but **not Haar meager** in  $X$ .

**Example 1.** Define the set  $A = \{(x_n)_{n \in \mathbb{N}} \in c_0 : \forall_{n \in \mathbb{N}} x_n \geq 0\}$ . Such set is a closed nowhere dense set (see [6, Example 6.2]), which belongs to the filter  $\mathcal{F}(c_0)$ .

Let  $S := B \times A \subset c_0$ , where  $B \subset \mathbb{R}$  is a meager set of the positive Lebesgue measure. Then the set  $S$  is Haar meager, but not Haar null in  $c_0$ .

Let  $T := C \times A \subset c_0$ , where  $C \subset \mathbb{R}$  is a comeager set of the Lebesgue measure zero. Such set  $T$  is Haar null, meager, but not Haar meager in  $c_0$ .

**Problem 2.** Let  $X$  be any abelian Polish group, which is not locally compact. How to find a Haar meager but not Haar null set and, conversely, how to construct a Haar null, meager but not Haar meager set in  $X$ ?

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